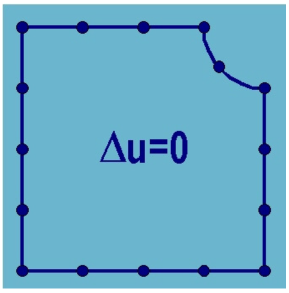

Boundary Element Methods

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Fundamental solution of the 2D Laplace equation

For simplicity the load-point is shifted in the origin. The derivation starts out by transforming the Laplace operator to polar coordinates (r, ϕ)

$$G_{,ii} = \frac{1}{r}\left(rG_{,r}\right)_{,r} + \frac{1}{r^2}G_{,\phi\phi}\,. \tag{154}$$

The excitation with the Dirac impulse is radial-symmetric and, since we are dealing with an infinite problem, there are no disturbances from the boundary, it is implied that the fundamental solution is radial-symmetric, too. Thus the last term in Eq. (154) vanishes. The Dirac impulse in polar coordinates is stated as $\delta(|x-\xi|) = \delta(r)/2\pi r$. Hence, a way to solve for G is to integrate Eq. (154). This yields

$$\begin{aligned}
\frac{1}{r}\left(rG_{,r}\right)_{,r} &= -\frac{\delta\left(r\right)}{2\pi r} \\
rG_{,r} &= -\int \frac{\delta\left(r\right)}{2\pi}\mathrm{dr} = -\frac{1}{2\pi} + C_1 \\
G_{,r} &= -\frac{1}{2\pi r} + \frac{C_1}{r} \\
G(r) &= -\frac{1}{2\pi}\ln r + C_1 \ln r + C_2
\end{aligned} \tag{155}$$

with the integration constants C_1 and C_2. It will be shown in the next section that $C_1 = 0$ for Eq. (155) being a valid solution of Eq. (152). The constant C_2 introduces the notion of a constant potential. It is arbitrary and is generally set to zero.

Verification of the impulse condition

The validity of a fundamental solution can be verified by evaluating the *impulse condition*. This condition carries out the integral over the partial differential equation over an arbitrary volume enclosing the Dirac impulse

$$\int_{\Omega_\infty} G_{,ii}\,\mathrm{d}\Omega = -\int_{\Omega_\infty} \delta(x)\,\mathrm{d}\Omega = -1\,. \tag{156}$$

Application of Gauss' theorem transforms the volume integral on the left to a surface integral

$$\int_{\Omega} G_{,ii}\,\mathrm{d}\Omega = \int_{\Gamma} G_{,i} n_i\,\mathrm{d}\Gamma \tag{157}$$

Since G is radial-symmetric, the gradient is also a pure function of the radius. In polar coordinates this reads as

$$G_{,i} = \begin{bmatrix} G_{,r} & 0 \end{bmatrix}^{\mathrm{T}} \tag{158}$$

Moreover, the outward normal on a circle is defined in polar coordinates as

$$\boldsymbol{n} = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{159}$$

Choosing the surface Γ as a circle of arbitrary radius leads to the impulse condition

$$\int_{\Gamma} G_{,r}\mathrm{d}\Gamma = -1\,. \tag{160}$$

Since G depends only on r and r is constant on a specific circle Γ, the impulse condition is reformulated as

$$\begin{aligned} G_{,r} \int\limits_{\Gamma} \mathrm{d}\Gamma &= -1 \\ G_{,r} \Gamma &= -1 \\ -\frac{1}{2\pi r} \, 2\pi r &= -1 \, , \end{aligned} \tag{161}$$

which proves that Eq. (155) is indeed a valid fundamental solution of Eq. (152). Note that this condition also implies the C_1 must be zero as stated before because otherwise the terms would not cancel to -1.

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